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# NONCRITICAL $W$ -STRINGS AND MINIMAL MODELS

Eric Bergshoeff

We perform a BRST analysis of the physical states described by a general noncritical  $W$ -string. A crucial feature of our analysis is that we introduce a special basis in the Hilbert space of physical states in which the BRST operator splits into a nested sum of nilpotent BRST operators. We argue that the cohomology of each nilpotent BRST operator occurring in the “nested” sum is closely related to a specific  $W$  minimal model. We discuss in detail the special case of the noncritical  $W_3$ -string.

## 1. INTRODUCTION

Noncritical strings are strings in which the two-dimensional gravitational fields do not decouple after quantisation. The non-decoupled gravity fields are called “Liouville” fields while the string coordinates are denominated as “matter” fields. The matter and Liouville fields form both *separately* as well as *together* realizations of the underlying Virasoro algebra. A BRST analysis of the physical states of the noncritical string in less than or precisely one dimension has been given in [1, 2].

It is natural to ask the question whether it is possible to extend the Virasoro symmetries and construct corresponding new (noncritical) string theories [3]. It turns out that this is indeed possible. The extended Virasoro symmetries are referred to as  $W$ -symmetries (for a review, see [4]). A special feature of the  $W$ -symmetries is that they form so-called nonlinear algebras. This makes their analysis nontrivial. Noncritical string theories consistent with  $W$ -symmetries are called noncritical  $W$ -strings [5]. The hope is that due to their extended symmetry structure  $W$ -strings have special properties which make them for specific applications preferable over ordinary string theories. Not much is known about the properties of  $W$ -strings. For instance, we don’t know yet what the physical states are of a general noncritical  $W$ -string. Clearly, in order to investigate whether  $W$ -strings can be an alternative for ordinary strings, we have to understand what the precise physical states are. In this talk I will report on recent results we obtained that help in clarifying some of the structure of the physical states. The work I will describe is based on references [6,7] and [8].

In a noncritical  $W$ -string the matter and Liouville fields *separately* form a realization of the underlying  $W$ -algebra. At first sight, one would expect that *together* they would also form a realization of the underlying nonlinear  $W$ -algebra, as in the case of the ordinary noncritical string. However, it turns out that, due to the nonlinear nature of the algebra, the matter and Liouville fields *together* cannot realize the same  $W$ -algebra. A way out of this apparent inconsistency was given in [9] where it was shown that nevertheless a nilpotent BRST operator for the matter + Liouville fields could be constructed. This is made possible by the fact that at the classical level the sums of the generators of the  $W$  algebras in the matter and Liouville sectors still form a closed Poisson-bracket algebra, albeit with field-dependent structure functions [10]<sup>1</sup>. This algebra differs from the original  $W$ -algebra. For definiteness, we will call this algebra the “modified”  $\tilde{w}$ -algebra.<sup>2</sup>

Having a nilpotent BRST operator at one’s disposal, one can proceed with a calculation of the cohomology and thus the spectrum of the noncritical  $W$ -string. Recent results about the spectrum of the noncritical  $W_3$  string have been given in [9, 12]. Based on a counting argument, on the basis of comparing values of central charges, it has been suggested that there is a relationship between the physical spectrum of a critical  $W$ -string<sup>3</sup> and Virasoro minimal models [13]. This relation can be extended to include other  $W$  minimal models as well [14, 15]. In the case of the critical  $W_3$ -string the relation has been confirmed and been made more explicit in a series of recent papers [15–18]. The relevant minimal model in this case is given by the  $c = 1/2$  Ising model.

<sup>1</sup> Alternatively, the existence of this BRST operator can be understood from the covariant action for  $W$  gravity coupled to matter [11].

<sup>2</sup> We denote the quantum extension of a classical  $w$ -algebra, provided it exists, by a  $W$ -algebra.

<sup>3</sup> We use here the nomenclature of [9]. Note that the “critical”  $W$ -string can be obtained from the “noncritical”  $W$ -string by setting the Liouville fields equal to zero.

It turns out that in the critical case the analysis of the cohomology can be simplified by going to a special basis in the Hilbert space [16]. Recently, we have shown that the same is true for the noncritical case [7]. It turns out that the new basis is very convenient for describing the relation between  $W$ -strings and minimal models. We will describe the situation for the noncritical  $W$ -string in more detail in the next section.

## 2. NONCRITICAL $W$ -STRINGS AND MINIMAL MODELS

We restrict the discussion to the so-called  $w_N$ -algebra [19] and assume that the matter and Liouville fields are realised in terms of  $N - 1$  free scalars  $\phi_k$  and  $\sigma_k$  ( $k = 1, \dots, N - 1$ ), respectively. It turns out that one can always perform a canonical transformation in the matter sector such that the “modified”  $\tilde{w}_N$ -algebra manifestly has a “nested” set of subalgebras

$$v_N^N \subset v_N^{N-1} \subset \dots \subset v_N^2 \equiv \tilde{w}_N, \quad (2.1)$$

where the subalgebra  $v_N^n$  consists of generators of spin  $s = \{n, n + 1, \dots, N\}$ , respectively. Each generator of spin  $s = n$  depends on  $N - n + 1$  of the  $N - 1$  matter scalars and all the  $N - 1$  Liouville scalars. In the new basis the BRST charge  $Q_N$  of the  $\tilde{w}_N$ -algebra has the following nested structure:

$$Q_N^N \subset Q_N^{N-1} \subset \dots \subset Q_N^2 \equiv Q_N, \quad (2.2)$$

where  $Q_N^n$  is the BRST charge corresponding to the subalgebra  $v_N^n$ . The BRST charge  $Q_N^n$  depends on  $N - n + 1$  matter scalars, all the  $N - 1$  Liouville scalars and the ghost and antighost fields of the spin  $n, \dots, N$  symmetries. The inclusion symbols in (2.2) indicate that the BRST charge  $Q_N^n$  can be obtained from the BRST charge  $Q_N^{n-1}$  by setting in the expression for  $Q_N^{n-1}$  the ghosts corresponding to the spin- $(n - 1)$  symmetries equal to zero. All these classical properties survive quantisation where the BRST charges become operators. To distinguish between BRST operators and BRST charges, we will write the operators with boldface.

It is instructive to consider the central charge contributions of the different fields in the new basis. The central charge contribution of the ghosts and antighosts of the spin  $s = k$  symmetry, the Liouville fields, and the matter fields are given by

$$\begin{aligned} c_{gh}(k) &= -2(6k^2 - 6k + 1), \\ c_l &= (N - 1) \left\{ 1 - 4(Q^2 - N(N - 1)) \frac{N + 1}{N - 1} \right\}, \\ c_m &= (N - 1) \left\{ 1 + 4Q^2 \frac{N + 1}{N - 1} \right\}, \end{aligned} \quad (2.3)$$

where  $Q$  is the background of the matter scalar  $\phi_{N-1}$ . Adding up the contributions of the spin  $s = 2, \dots, s = N$  ghosts we find

$$c_{gh} = \sum_{k=2}^N c_{gh}(k) = -2(N - 1)(2N^2 + 2N + 1). \quad (2.4)$$

Note that we have

$$c_m + c_l + c_{gh} = 0, \quad (2.5)$$

as is required to allow for a nilpotent BRST operator.

We now choose the Liouville sector corresponding to a  $(p, q)$  minimal model of the  $W_N$ -algebra. This requires that we take for the background charge  $Q$  one of the following values:

$$Q_{\min}^2 = N(N - 1) \frac{(p + q)^2}{4pq}, \quad (2.6)$$

where  $p$  and  $q$  are non-negative integers which are relative prime. The central charge contribution of the Liouville sector is then given by

$$c_l^{(p,q)} = (N - 1) \left\{ 1 - \frac{(p - q)^2}{pq} N(N + 1) \right\}. \quad (2.7)$$

Given this choice of the Liouville sector, we now count the central charge contribution of the fields that are present in the  $\mathbf{Q}_N^n$  BRST operator, for fixed  $n$ . The  $N - n + 1$  matter scalars present in  $\mathbf{Q}_N^n$  contribute

$$c_m^n = \sum_{k=n-1}^{N-1} (1 + 12(\alpha_k)^2), \quad (2.8)$$

where  $\alpha_k$  is the background charge of the matter scalar  $\phi_k$ :

$$\alpha_k = Q_{\min} \sqrt{\frac{k(k+1)}{N(N-1)}}. \quad (2.9)$$

Similarly, the central charge contribution of the spin  $s = n, \dots, s = N$  ghosts contribute

$$c_{gh}^n = -2 \sum_{k=n}^N (6k^2 - 6k + 1). \quad (2.10)$$

Denoting the central charge contributions of the fields occurring in  $\mathbf{Q}_N^n$ , for a given choice of  $p, q$  and  $n$ , by  $c_N^{n;(p,q)}$  we find

$$c_N^{n;(p,q)} = c_l^{(p,q)} + c_m^n + c_{gh}^n = (n-2) \left\{ (2n-1)^2 - n(n-1) \frac{(p+q)^2}{pq} \right\}, \quad (2.11)$$

which is exactly the central charge corresponding to the  $(p, q)$  minimal model of the  $W_{n-1}$ -algebra!

The above counting argument suggests a relation between the cohomology of the  $\mathbf{Q}_N^n$  BRST operator and the  $(p, q)$  minimal model of the  $W_{n-1}$ -algebra. Note that the critical case corresponds in the above discussion to the special case in which we choose the Liouville sector to correspond to the  $(N+1, N)$   $W_N$  minimal model with  $c_l^{(N+1,N)} = 0$ . In that case the central charge corresponding to the  $\mathbf{Q}_N^n$  operator is given by

$$c_N^{n;(N+1,N)} = (n-2) \left\{ 1 - \frac{n(n-1)}{N(N+1)} \right\}, \quad (2.12)$$

which is the  $(N+1, N)$  minimal model of the  $W_{n-1}$ -algebra. For instance, for  $n = N = 3$  we find the  $c_3^{3;(4,3)} = 1/2$  Ising model.

Another interesting special case occurs if we choose the Liouville sector corresponding to the  $(n, n-1)$  minimal model of the  $W_N$ -algebra with central charge  $(2 \leq n \leq N)$ :

$$c_l^{(n,n-1)} = (N-1) \left\{ 1 - \frac{N(N+1)}{n(n-1)} \right\}. \quad (2.13)$$

In that case we find that  $c_N^{n;(n,n-1)} = 0$ . This means that we effectively end up with a “critical”  $W_{n-1}$ -string. For instance, for  $n = N = 3$  we have  $c_l^{(3,2)} = -2$ . In that case the central charge contribution of  $\sigma_2$  is zero and the Liouville sector can be realized in terms of the *single* scalar  $\sigma_1$  [20]. The central charge contribution of the matter scalar  $\phi_1$  now equals 26. In the basis we are using (see the next section)  $\phi_1$  only occurs via its energy momentum tensor and we can replace it by 26 scalars  $X^\mu$  ( $\mu = 0, 1, \dots, 25$ ). We thus end up effectively with an ordinary “critical” string theory [10].

In the next section we will consider the special case of the noncritical  $W_3$ -string in more detail.

### 3. THE NONCRITICAL $W_3$ -STRING

We consider a realisation of the noncritical  $W_3$ -string in terms of two matter scalars  $\phi_1, \phi_2$  and two Liouville scalars  $\sigma_1, \sigma_2$ . We furthermore introduce spin-two ghosts  $(c, b)$  and spin-three ghosts  $(\gamma, \beta)$ . The ghosts  $c, \gamma$  ( $b, \beta$ ) have ghost number  $+1$  ( $-1$ ). The conformal weights of  $(c, b)$  and  $(\gamma, \beta)$  are  $(-1, 2)$  and  $(-2, 3)$ , respectively. The background charges  $Q_{\phi_k}, Q_{\sigma_k}$  ( $k = 1, 2$ ) of the matter and Liouville scalars, respectively, are given by

$$\begin{aligned} Q_{\phi_1} &= \frac{1}{3} \sqrt{3} Q, \\ Q_{\phi_2} &= Q, \\ Q_{\sigma_1} &= \frac{1}{3} \sqrt{3} \sqrt{(6 - Q^2)}, \\ Q_{\sigma_2} &= \sqrt{6 - Q^2}. \end{aligned} \quad (3.1)$$

The relative coefficient between  $Q_{\phi_1}$  and  $Q_{\phi_2}$  is chosen such that in the realisation of the  $W_3$ -algebra by the matter sector separately the spin-2 and spin-3 generators contain  $\phi_1$  only via its energy momentum tensor [21]. The central charge contributions of the matter, Liouville fields, spin-2 ghosts  $(c, b)$ , and spin-3 ghosts  $(\gamma, \beta)$  are given by

$$\begin{aligned} c_{\phi_1} &= 1 + 4Q^2, \\ c_{\phi_2} &= 1 + 12Q^2, \\ c_{\sigma_1} &= 25 - 4Q^2, \\ c_{\sigma_2} &= 73 - 12Q^2, \\ c_{(c,b)} &= -26, \\ c_{(\gamma,\beta)} &= -74. \end{aligned} \tag{3.2}$$

Note that the total central charge contribution equals zero.

In [7] the BRST-operator of the noncritical  $W_3$ -string in the new basis, discussed in the previous section, was calculated. It is given by  $Q = \oint j$ , with  $j = j_0 + j_1$  given by

$$\begin{aligned} j_0 &= c \{ T_M + T_L + T_{(\gamma,\beta)} + \frac{1}{2} T_{(c,b)} \}, \\ j_1 &= \gamma \left[ \frac{i}{3\sqrt{6}} \{ 4(\partial\varphi_2)^3 - 12Q\partial\varphi_2\partial^2\varphi_2 + (-15 + 4Q^2)\partial^3\varphi_2 \} \right. \\ &\quad \left. + i \{ W_L - \frac{2}{\sqrt{6}}\partial\varphi_2 T_L + \frac{1}{\sqrt{6}}Q\partial T_L \} \right. \\ &\quad \left. - i\sqrt{6} \{ \partial\varphi_2\partial\gamma\beta + \frac{1}{3}Q\partial\beta\partial\gamma \} \right]. \end{aligned} \tag{3.3}$$

In the new basis the BRST operator  $Q \equiv Q_3$  decomposes as

$$Q = Q_0 + Q_1, \tag{3.4}$$

where  $Q_0 = \oint j_0$  and  $Q_1 = \oint j_1$  are separately nilpotent and anticommute. The nilpotent BRST operator  $Q_0$  corresponds to the Virasoro subalgebra.

A crucial feature of the new basis is that the scalar  $\phi_1$  and the spin-2 ghosts  $(c, b)$  are absent in  $j_1$ . Note also that the Liouville fields only occur in  $j_1$  via their Liouville generators

$$\begin{aligned} T_L &= T_{\sigma_1} + T_{\sigma_2} = -\frac{1}{2}(\partial\sigma_1)^2 + \frac{1}{3}\sqrt{3(6-Q^2)}\partial^2\sigma_1 - \frac{1}{2}(\partial\sigma_2)^2 + \sqrt{6-Q^2}\partial^2\sigma_2, \\ W_L &= \frac{1}{18}i\sqrt{6} \left\{ (\partial\sigma_2)^3 - 3\sqrt{6-Q^2}\partial\sigma_2\partial^2\sigma_2 \right. \\ &\quad \left. + (6-Q^2)\partial^3\sigma_2 + 6\partial\sigma_2 T_{\sigma_1} - 3\sqrt{6-Q^2}\partial T_{\sigma_1} \right\}. \end{aligned} \tag{3.5}$$

Here  $T_L$  and  $W_L$  satisfy the standard  $W_3$ -algebra. Finally, in  $j_0$  we find, besides  $T_L$ , the energy-momentum tensors of the matter and ghost sectors:

$$\begin{aligned} T_M &= T_{\phi_1} + T_{\phi_2} = -\frac{1}{2}(\partial\phi_1)^2 + \frac{1}{3}\sqrt{3}Q\partial^2\phi_1 - \frac{1}{2}(\partial\phi_2)^2 + Q\partial^2\phi_2, \\ T_{(c,b)} &= -2b\partial c - (\partial b)c, \\ T_{(\gamma,\beta)} &= -3\beta\partial\gamma - 2(\partial\beta)\gamma. \end{aligned} \tag{3.6}$$

For the convenience of the reader we will give some of the formulae of the previous section specifically for  $N = 3$ . The total Liouville central charge  $c_l$  and  $c_3^3$ , the contribution to the central charge of the fields that play a role in  $Q_1$  ( $\phi_2$ , the spin-3 ghosts, and the Liouville scalars) are given by:

$$\begin{aligned} c_l &= 98 - 16Q^2, \\ c_3^3 &= 25 - 4Q^2. \end{aligned} \tag{3.7}$$

We now choose for the Liouville sector a  $(p, q)$   $W_3$  minimal model. We therefore take the background charge parameter  $Q$  equal to<sup>4</sup>

$$Q_{\min} = \frac{3(q+p)}{\sqrt{6pq}}, \quad \sqrt{6-Q_{\min}^2} = \frac{3i(q-p)}{\sqrt{6pq}}. \quad (3.8)$$

With this choice of  $Q$  we find the following values for  $c_l$  and  $c_3^3$ :

$$\begin{aligned} c_l^{(p,q)} &= 2\left(1 - \frac{12(p-q)^2}{pq}\right), \\ c_3^{3;(p,q)} &= 1 - \frac{6(p-q)^2}{pq}. \end{aligned} \quad (3.9)$$

The values  $c_l^{(p,q)}$  and  $c_3^{3;(p,q)}$  correspond to the central charges of the  $(p, q)$   $W_3$  minimal model, and the  $(p, q)$  Virasoro minimal model, respectively. In the next section we will discuss the cohomology of  $\mathbf{Q}_1$  and make the relation with the Virasoro minimal models more explicit.

#### 4. THE $\mathbf{Q}_1$ COHOMOLOGY

To keep the discussion simple we restrict ourselves in this section to the special case in which  $(p, q) = (4, 3)$ . In that case  $c_3^{3;(4,3)} = 1/2$  so that the Virasoro minimal model corresponds to the Ising model. A more detailed discussion of the situation for general values of  $(p, q)$  can be found in [8]. Our purpose here is to merely illustrate some of the features of the general case.

For  $(p, q) = (4, 3)$  the Liouville sector describes the trivial  $W_3$  minimal model which has  $c_l^{(4,3)} = 0$ . Therefore this situation reduces to the critical  $W_3$ -string discussed in [15–18].<sup>5</sup> The value of  $Q_{\min}$  in this case is

$$\sqrt{72} Q_{\min} = 21, \quad \sqrt{72} \sqrt{6-Q_{\min}^2} = -3i. \quad (4.1)$$

Furthermore the background charges are given by

$$\sqrt{72} (Q_{\phi_2}, Q_{\sigma_1}, Q_{\sigma_2}) = (21, -\sqrt{3}i, -3i). \quad (4.2)$$

Our goal is to identify the primary operators of the Ising model. For this it turns out to be sufficient to consider the states at lowest ghost number at level 0 and level 1. We first consider the level 0 states. The states of lowest ghost number are of the form<sup>6</sup>

$$\mathbf{V}_0(p_2, s_1, s_2) = (\partial\gamma)\gamma e^{ip_2\phi_2 + is_1\sigma_1 + is_2\sigma_2}. \quad (4.3)$$

The condition  $\mathbf{Q}_1 \mathbf{V}_0(p_2, s_1, s_2) = 0$  determines the momenta of the three fields. The resulting cubic equation factorizes, and we obtain the following three solutions of  $p_2$  in terms of  $s_1$  and  $s_2$ :

$$\begin{aligned} (A_0) \quad & \sqrt{72} p_2 = i\{\sqrt{72} s_2 - 18\}, \\ (E_0) \quad & \sqrt{72} p_2 = i\left\{\frac{1}{2}\sqrt{72}(\sqrt{3}s_1 - s_2) - 21\right\}, \\ (C_0) \quad & \sqrt{72} p_2 = i\left\{-\frac{1}{2}\sqrt{72}(\sqrt{3}s_1 + s_2) - 24\right\}. \end{aligned} \quad (4.4)$$

There are further restrictions on the values of  $s_1$  and  $s_2$  which follow from the requirement that the Liouville sector should describe a trivial  $W_3$  minimal model with  $c_l^{(4,3)} = 0$ . These restrictions are given by (see, e.g., [4])

$$\begin{aligned} \sqrt{72} s_1 &= -3\sqrt{3}r_2, \\ \sqrt{72} s_2 &= -6r_1 - 3r_2, \end{aligned} \quad (4.5)$$

<sup>4</sup>Note that  $Q_{\min}$  is determined up to sign by (2.6). In (3.8) we have made a specific choice for this sign.

<sup>5</sup>We should note that in the generic case the Liouville scalars do contribute to the central charge and cannot be set equal to zero.

<sup>6</sup>The level of a state is defined as follows: level  $\equiv h + 3$ , where  $h$  is the weight of the fields in front of the exponent. For instance, the level of the state (4.3) is given by  $(-1 - 2) + 3 = 0$ . The state (4.3) is of lowest ghost number. The other level zero states whose ghost numbers are one higher have a factor  $(\partial^2\gamma)(\partial\gamma)\gamma$  in front of the exponent.

with

$$(r_1, r_2) = (0, 0), (0, 1), \text{ or } (1, 0). \quad (4.6)$$

The contribution  $h_l$  of the Liouville sector to the total conformal weight  $h_{Vir}$  can be calculated by using the generic formula in which the conformal weight of a vertex operator  $\exp(ip_\phi\phi)$ , for a scalar  $\phi$  with background charge  $Q_\phi$ , is given by

$$h_\phi = \frac{1}{2}(p_\phi + iQ_\phi)^2 + \frac{1}{2}Q_\phi^2. \quad (4.7)$$

As expected, we find that for all three Liouville states (4.6) we have  $h_l = 0$ . Hence there is a threefold degeneracy. In fact, all the states are connected by the following  $Z_3$  transformation which leaves the Liouville weight invariant:

$$(r_1, r_2) \rightarrow (1 - r_1 - r_2, r_1) \rightarrow (r_2, 1 - r_1 - r_2) \rightarrow (r_1, r_2). \quad (4.8)$$

In general there can be even a sixfold degeneracy due to the following  $Z_2$  transformation:

$$(r_1, r_2) \rightarrow (r_2, r_1) \rightarrow (r_1, r_2). \quad (4.9)$$

However, since the state  $(r_1, r_2) = (0, 0)$  is invariant under this  $Z_2$  transformation we have a threefold degeneracy only. In principle we can identify the three states given by (4.6): the  $c_l^{(4,3)} = 0$   $W_3$  minimal model has only one primary state with weight  $h_l = 0$ . For instance, we could choose the  $(r_1, r_2) = (0, 0)$  state to represent the *single* primary state. In that case the Liouville fields effectively become zero and our calculation would become identical to that of [15–18]. However, in order to mimic as much as possible the generic situation where the Liouville fields cannot be set equal to zero we will not do so. We will keep all three states given by (4.6). In the next section we will see how the theory automatically takes care of the threefold degeneracy. Note that every Liouville state *a priori* gives rise to *three* states in the  $\mathbf{Q}_1$  cohomology due to the fact that we have the *three* solutions  $(A_0)$ ,  $(B_0)$ , and  $(C_0)$ . So in total we have 9 states with the total conformal weights and momenta<sup>7</sup> given in Table 1.

Table 1  
Momenta and conformal weights for the states  $\mathbf{V}_{0,A_0}$ ,  $\mathbf{V}_{0,B_0}$ , and  $\mathbf{V}_{0,C_0}$  given in (4.4).

| Level Zero States of Lowest Ghost Number |              |                            |                |
|--|--------------|----------------------------|----------------|
| Solution                                 | $(r_1, r_2)$ | $\sqrt{72}(p_2, s_1, s_2)$ | $h_{Vir}$      |
| $(A_0)$                                  | (0,0)        | (-18i,0,0)                 | 0              |
|  | (0,1)        | (-21i, -3√3, -3)           | $\frac{1}{16}$ |
|  | (1,0)        | (-24i,0,-6)                | 0              |
| $(B_0)$                                  | (0,0)        | (-21i,0,0)                 | $\frac{1}{16}$ |
|  | (0,1)        | (-24i, -3√3, -3)           | 0              |
|  | (1,0)        | (-18i,0,-6)                | 0              |
| $(C_0)$                                  | (0,0)        | (-24i,0,0)                 | 0              |
|  | (0,1)        | (-18i, -3√3, -3)           | 0              |
|  | (1,0)        | (-21i,0,-6)                | $\frac{1}{16}$ |

The conformal weights of the primary operators of the  $c = 1/2$  Ising model are given by

$$h_{Vir}(r, t) = \frac{1}{48}((3(r+1) - 4(t+1))^2 - 1), \quad (4.10)$$

<sup>7</sup>Note that the ghost factor  $(\partial\gamma)\gamma$  contributes  $-3$  to the total conformal weight.

with  $0 \leq r \leq 2, 0 \leq t \leq 1$ . This leads to six operators. However, the Virasoro weights are invariant under the following  $\mathbb{Z}_2$  transformation:

$$(r, t) \rightarrow (2 - r, 1 - t) \rightarrow (r, t), \quad (4.11)$$

which leads to a twofold degeneracy. For instance, taking  $t = 0$ , we find three states with weights

$$h_{Vir}(0, 0) = 0, \quad h_{Vir}(1, 0) = \frac{1}{16}, \quad h_{Vir}(2, 0) = \frac{1}{2}. \quad (4.12)$$

Comparing Table 1 with (4.12) we conclude that, besides the expected threefold degeneracy, we find twice too many states with weight 0, while we are missing a state with weight  $\frac{1}{2}$ .

To find the missing operator with weight  $1/2$  we consider the following operators at level one of lowest ghost number:

$$\mathbf{V}_1(p_2, s_1, s_2) = \gamma e^{ip_2\phi_2 + is_2\sigma_1 + is_2\sigma_2}. \quad (4.13)$$

The condition  $\mathbf{Q}_1 \mathbf{V}_1(p_2, s_1, s_2) = 0$ , taken together with the restrictions on the Liouville momenta given in (4.5), now leads to the two solutions [8]<sup>8</sup> given in Table 2.

Table 2  
Momenta and conformal weights for the states  $\mathbf{V}_{1,A_1}$   
and  $\mathbf{V}_{1,B_1}$  of level one of lowest ghost number.

| Level One States of Lowest Ghost Number |              |                            |                |
|---|--------------|----------------------------|----------------|
| Solution                                | $(r_1, r_2)$ | $\sqrt{72}(p_2, s_1, s_2)$ | $h_{Vir}$      |
| $(A_1)$                                 | (0,0)        | (-12i, 0, 0)               | $\frac{1}{2}$  |
|   | (1,0)        | (-9i, 0, -6)               | $\frac{1}{16}$ |
| $(B_1)$                                 | (0,0)        | (-9i, 0, 0)                | $\frac{1}{16}$ |
|   | (0,1)        | (-12i, -3 $\sqrt{3}$ , -3) | $\frac{1}{2}$  |

We indeed find the missing operator of weight  $1/2$ . Combining Tables 1 and 2 we conclude that we have obtained all the primary operators of the  $c = 1/2$  Ising model at level 0 and 1 of the  $\mathbf{Q}_1$  cohomology. However, instead of finding exactly three primary operators of the desired weights, we obtain  $9 + 4 = 13$  states. Even using the expected  $\mathbb{Z}_3$  degeneracy at level 0, i.e., choosing only the states with  $(r_1, r_2) = 0$ , we are still left with  $3 + 2 = 5$  states, which are too many. In the next section we will show how the number of states can be reduced to exactly the three operators of the Ising model.

## 5. SCREENING AND PICTURE-CHANGING OPERATORS

Following the case of the critical string [15–18], we conjecture that all states in the  $\mathbf{Q}_1$  cohomology are related to the three basic operators of the Ising model by the action of picture-changing and screening operators or are descendants of these primary operators<sup>9</sup>. In this section we will confirm this conjecture and show that the states we have obtained in the previous section are all connected to each other such that we are left with three independent states of weight 0,  $\frac{1}{16}$ , and  $\frac{1}{2}$  which can be identified with the three basic operators of the Ising model.

Screening operators (anti-)commute with  $\mathbf{Q}_1$  and have weight 0. They transform physical states into other physical states, provided the action of such an operator on a state is welldefined (see [18] and the discussion below). Screening operators may change the level.

<sup>8</sup>Note that the ghost factor  $\gamma$  contributes  $-2$  to the total conformal weight.

<sup>9</sup>This concerns the states with continuous momenta. The situation of the discrete states is more complicated.



Picture-changing operators are of the form  $[\mathbf{Q}, \phi]$ , where  $\phi$  is a scalar field. They also produce solutions of the cohomology when acting on a physical state. In our case there are three such operators, corresponding to  $\phi_2$ ,  $\sigma_1$ , and  $\sigma_2$ . They have weight 0, change the ghost number of a physical state by a single unit but do not change the level. Applying the same picture-changing operator twice gives 0.

Let us first give a list of the screening and picture-changing operators we will need for our discussion (for a more complete list, see [8]). There are two screening operators which contain both matter as well as Liouville scalars:

$$\begin{aligned} S_1 &= \oint dz \beta(z) e^{\frac{i}{\sqrt{72}} \{6i\phi_2(z) - 3\sqrt{3}\sigma_1(z) - 3\sigma_2(z)\}}, \\ S_2 &= \oint dz \beta(z) e^{\frac{i}{\sqrt{72}} \{8i\phi_2(z) + 8\sigma_2(z)\}}. \end{aligned} \quad (5.1)$$

There are four screening operators which involve only the Liouville fields:

$$\begin{aligned} T_1 &= \oint dz e^{\frac{i}{\sqrt{72}} \{6\sqrt{3}\sigma_1(z)\}}, \\ T_2 &= \oint dz e^{\frac{i}{\sqrt{72}} \{-8\sqrt{3}\sigma_1(z)\}}, \\ T_3 &= \oint dz e^{\frac{i}{\sqrt{72}} \{-3\sqrt{3}\sigma_1(z) + 9\sigma_2(z)\}}, \\ T_4 &= \oint dz e^{\frac{i}{\sqrt{72}} \{4\sqrt{3}\sigma_1(z) - 12\sigma_2(z)\}}. \end{aligned} \quad (5.2)$$

Finally, the three picture-changing operators are  $P_{\phi_2} = [\mathbf{Q}_1, \phi_2]$ ,  $P_{\sigma_1} = [\mathbf{Q}_1, \sigma_1]$ , and  $P_{\sigma_2} = [\mathbf{Q}_1, \sigma_2]$ . In this section we consider only the action of these picture-changing operators on the vacuum state (4.3). The only term which then contributes is the  $\partial^2 \gamma$ -contribution, which is present in all three picture-changing operators. It produces physical states

$$\bar{\mathbf{V}}_0(p_2, s_1, s_2) = (\partial^2 \gamma)(\partial \gamma) \gamma e^{ip_2 \phi_2 + is_1 \sigma_1 + is_2 \sigma_2}, \quad (5.3)$$

for the same values of the momenta obtained in Table 1. Since all picture-changing operators act the same way in this case, we will denote them collectively by  $P$ :

$$\begin{aligned} P &= -\frac{i}{3\sqrt{6}} \left[ 12\gamma(\partial\varphi_2)^2 + 12Q(\partial\gamma)\partial\varphi_2 + (-15 + 4Q^2)\partial^2\gamma \right] \\ &\quad + \frac{2}{\sqrt{6}}\gamma T_L + i\sqrt{6}\gamma(\partial\gamma)\beta. \end{aligned} \quad (5.4)$$

We now explain the action of the screening operators. All our screening operators are of the form

$$S_i = \oint dz_i K_i(\beta(z_i), \gamma(z_i)) \exp(i \sum_m p_{m, S_i} \phi_m(z_i)), \quad (5.5)$$

where  $\phi_m$  is a set of scalar fields, and  $p_{m, S_i}$  are the screening momenta in the operator  $S_i$  for the  $m$ -th field. These operators act on states of the form

$$\mathbf{O} = L(\beta(w), \gamma(w)) \exp(i \sum_m p_m \phi_m(w)). \quad (5.6)$$

The condition under which this action is welldefined is discussed in detail in [18]. This condition arises from the fact that one requires that the successive OPE's of the screening operators  $S_1, \dots, S_n$  with  $\mathbf{O}$  give rise to a single factor  $(z_1 - w)^{P_n}$ , where  $P_n$  must be an integer. The OPE's of the ghost contributions will similarly give a factor  $(z_1 - w)^{P_{gh}}$ , where  $P_{gh}$  is *guaranteed* to be an integer. The last integral over  $z_1$  then gives a welldefined and non-trivial result if

$$P_n + P_{gh} = -1. \quad (5.7)$$

In practice we always deal with a situation where the screening operators connect two states  $\mathbf{O}$  and  $\mathbf{O}' = S_1 \dots S_n \mathbf{O}$  which have *equal* conformal weight while the momenta of the screening operators always interpolate between those of  $\mathbf{O}$  and  $\mathbf{O}'$ , i.e.,  $p'_m = p_m + \sum_i p_{m, S_i}$ . One can show that for this situation  $P_n$  is *always* an integer given by [8]

$$P_n = h_{L, \mathbf{O}} - h_{L, \mathbf{O}'} - 1 + \sum_i h_{K_i}, \quad (5.8)$$

where  $h_{K_i}$ ,  $h_{L, \mathbf{O}}$ ,  $h_{L, \mathbf{O}'}$  are the conformal weights of the ghost contributions to  $S_i$ ,  $\mathbf{O}$ , and  $\mathbf{O}'$ , respectively. One then still has to satisfy the condition (5.7). In the following we will see that this condition can also be satisfied by the use of appropriate picture-changing operators (see the discussion after Eq. (5.12)).

In discussing the action of the screening operators, it is useful to characterize their effect on the momenta and on the labels  $(r_1, r_2)$  in the Liouville sector. Consider a screening operator

$$S = (S_1)^{m_1} (S_2)^{m_2} (T_1)^{n_1} (T_2)^{n_2} (T_3)^{n_3} (T_4)^{n_4}. \quad (5.9)$$

The changes in the momenta due to (5.9) are given by

$$\begin{aligned} \sqrt{72} \Delta p_2 &= i(6m_1 + 8m_2), \\ \sqrt{72} \Delta s_1 &= \sqrt{3}(-3m_1 + 6n_1 - 8n_2 - 3n_3 + 4n_4), \\ \sqrt{72} \Delta s_2 &= -3m_1 + 8m_2 + 9n_3 - 12n_4, \end{aligned} \quad (5.10)$$

with  $p'_m = p_m + \Delta p_m$ . The change in the Liouville momenta induces a change in the labels  $(r_1, r_2)$  in (4.5) given by

$$\begin{aligned} \Delta r_1 &= -\frac{4}{3}m_2 + n_1 - \frac{4}{3}n_2 - 2n_3 + \frac{8}{3}n_4, \\ \Delta r_2 &= m_1 - 2n_1 + \frac{8}{3}n_2 + n_3 - \frac{4}{3}n_4. \end{aligned} \quad (5.11)$$

Now that the action of the screening operators has been clarified, let us first discuss the way they act on the states at level 0 given in Table 1. We found for all three solutions  $(A_0)$ ,  $(B_0)$ , and  $(C_0)$  *two* states of weight zero. It turns out that these two states are connected to each other by the action of the following screening and picture-changing operators:

$$\begin{aligned} \mathbf{V}_{0, (A_0)}(0, 0) &= S_1 T_1 T_3 P \mathbf{V}_{0, (A_0)}(1, 0), \\ \mathbf{V}_{0, (B_0)}(1, 0) &= S_1 T_1 P \mathbf{V}_{0, (B_0)}(0, 1), \\ \mathbf{V}_{0, (C_0)}(0, 1) &= S_1 P \mathbf{V}_{0, (C_0)}(0, 0). \end{aligned} \quad (5.12)$$

Note that the above action of screening operators gives, following Eq. (5.8),  $P_n = (-3) - (-3) - 1 + 3 = 2$ . According to Eq. (5.7) we therefore need  $P_{gh} = -3$ . This is achieved by the addition of the picture-changing operator  $P$  in Eq. (5.12). The ghost structure of  $P\mathbf{O}$  is now given by  $(\partial^2 \gamma)(\partial \gamma)\gamma$ . The OPE of the ghost  $\beta$  in  $S_1$  with the  $(\partial^2 \gamma)$  factor in  $(\partial^2 \gamma)(\partial \gamma)\gamma$  gives  $P_{gh} = -3^{10}$ .

We are still left with three independent states of weight 0 at level zero, which is too many. It turns out that the weight zero states in the  $(B_0)$  and  $(C_0)$  solutions are connected to the one in the  $(A_0)$  solutions as follows:

$$\begin{aligned} \mathbf{V}_{0, (B_0)}(1, 0) &= T_1 T_2 (T_3)^2 (T_4)^2 \mathbf{V}_{0, (A_0)}(0, 0), \\ \mathbf{V}_{0, (C_0)}(0, 1) &= (T_1)^2 (T_2)^2 T_3 T_4 \mathbf{V}_{0, (A_0)}(0, 0). \end{aligned} \quad (5.13)$$

Finally, we are left with three states at level zero of weight 1/16. The ones in the  $(B_0)$  and  $(C_0)$  solution are related to the weight 1/16 state in the  $(A_0)$  solution in the following way:

$$\begin{aligned} \mathbf{V}_{0, (B_0)}(0, 0) &= (T_1)^2 T_2 (T_3)^3 (T_4)^2 \mathbf{V}_{0, (A_0)}(0, 1), \\ \mathbf{V}_{0, (C_0)}(1, 0) &= (T_1)^3 (T_2)^2 T_3 T_4 \mathbf{V}_{0, (A_0)}(0, 1). \end{aligned} \quad (5.14)$$

<sup>10</sup>The OPE's of  $\beta$  with the other ghost factors in  $(\partial^2 \gamma)(\partial \gamma)\gamma$  give rise to less singular terms.

Note that the action of the screening operators in Eqs. (5.13) and (5.14) immediately gives  $P_n = (-3) - (-3) - 1 + 0 = -1$  and hence there is no need to insert a picture-changing operator.

We now consider the action of the screening and picture-changing operators on the states at level one. First of all the two states of weight  $1/16$  are connected to weight  $1/16$  states at level zero and therefore do not represent new states:

$$\begin{aligned} \mathbf{V}_{1,(A_1)}(1,0) &= (S_1)^2 T_1 P \mathbf{V}_{0,(B_0)}(0,0) , \\ \mathbf{V}_{1,(B_1)}(0,0) &= (S_1)^2 (T_1)^2 T_3 P \mathbf{V}_{0,(A_0)}(0,1) . \end{aligned} \quad (5.15)$$

Note that we now have  $P_n = (-3) - (-2) - 1 + 6 = +4$ . The two  $\beta$  ghosts in  $(S_1)^2$  contract with the  $(\partial^2 \gamma)(\partial \gamma)$  factor of the  $(\partial^2 \gamma)(\partial \gamma)\gamma$  term in  $PO$  to give  $P_{gh} = -5$  so that the condition (5.7) again is satisfied.

Our final task is to show that the two weight  $1/2$  states at level one are connected to each other. This indeed turns out to be the case. The weight  $1/2$  state in the  $(B_1)$  solution is related to the one in the  $(A_1)$  solution by the action of the following screening operators:

$$\mathbf{V}_{1,(B_1)}(0,1) = (T_1)^2 (T_2)^2 T_3 T_4 \mathbf{V}_{1,(A_1)}(0,0) . \quad (5.16)$$

In conclusion, we have shown that all the states we found in the previous section occur with multiplicity one, up to screening and picture-changing operators. We are thus left with three independent states of weight  $0$ ,  $1/16$ , and  $1/2$  which can be identified with the primary operators of the Ising model.

## 6. DISCUSSION

In this lecture we have discussed a relationship between noncritical  $W$ -strings and  $W$  minimal models. In the case of the  $W_N$ -algebra the relationship can be stated as follows. Suppose we restrict by hand the Liouville sector of the noncritical  $W$ -string to describe a  $(p, q)$  minimal model of the  $W_N$ -algebra. Consider the cohomology of the  $\mathbf{Q}_N^n$  operator defined in Section 2. This cohomology is now related to the primary operators of the  $(p, q)$  minimal model of the  $W_{n-1}$ -algebra in the sense that all states in the  $\mathbf{Q}_N^n$ -cohomology can be obtained from the primary operators of the  $(p, q)$  minimal model of the  $W_{n-1}$ -algebra by the action of screening and/or picture-changing operators or are descendants of these primary operators. We have not proven this relationship. In section 2 we have given a counting argument, based on comparing the values of the central charge contribution of the fields present in  $\mathbf{Q}_N^n$  with the corresponding  $(p, q)$  minimal model of the  $W_{n-1}$ -algebra. Furthermore in Sections 3, 4, and 5 we have shown in detail, for the non-critical  $W_3$  string, and for the special case where the Liouville sector describes a trivial  $W$  minimal model with central charge zero, how the states of lowest ghost number at level zero and one are indeed connected to the three primary operators of weight  $0$ ,  $1/16$ , and  $1/2$  of the Ising model. For more details about the generic case we refer to [8].

Finally, we note that our discussion did *not* include the cohomology of the complete BRST operator  $\mathbf{Q}_N^2 \equiv \mathbf{Q}_N$  of the non-critical  $W_N$ -string. Results about the cohomology of this complete BRST operator have been obtained for the noncritical  $W_3$  string [9, 12], the critical  $W_3$  string [22], and the so-called critical  $W_{2,4}$ -string [23]. We expect that the results discussed in this lecture will contribute to a better understanding of the complete cohomology. For a further discussion on the cohomology of the complete BRST operator we refer to [8].

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